

NONREPETITIVE SEQUENCES ON ARITHMETIC PROGRESSIONS

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ABSTRACT. A sequence $S = s_1 s_2 \dots s_n$ is *nonrepetitive* if no two adjacent blocks of S are identical. In 1906 Thue proved that there exist arbitrarily long nonrepetitive sequences over 3-element set of symbols. We study a generalization of nonrepetitive sequences involving arithmetic progressions. We prove that for every $k \geq 1$, there exist arbitrarily long sequences over at most $2k + 10\sqrt{k}$ symbols whose subsequences, indexed by arithmetic progressions with common differences from the set $\{1, 2, \dots, k\}$, are nonrepetitive. This improves a previous bound obtained in [9]. Our approach is based on a technique introduced recently in [12], which was originally inspired by a constructive proof of the Lovász Local Lemma due to Moser and Tardos [15]. We also discuss some related problems that can be attacked by this method.

1. INTRODUCTION

For a sequence $S = s_1 s_2 \dots s_n$ a *repetition* of size h is a block (subsequence of consecutive terms) of the form $XX = x_1 \dots x_h x_1 \dots x_h$. A sequence is *nonrepetitive* if it does not contain a repetition of any size $h \geq 1$. For example, the sequence 1231312 contains a repetition 3131 of size two, while 123132123 is nonrepetitive.

It is easy to see that the longest nonrepetitive sequence, which can be constructed over a set of two symbols, has length three. In 1906 Thue [18] proved, by a remarkable inductive construction, that there exist arbitrarily long nonrepetitive sequences over just three different symbols (see also [5], [4]). This discovery resulted in many unexpected applications inspiring a stream of research and leading to the emergence of new branches of mathematics with a variety of challenging open problems (see [1], [3], [8], [11], [14]).

One particular variant, proposed in [3], concerns *nonrepetitive tilings*, i.e., assignments of symbols to lattice points of the plane so that all lines in prescribed directions are nonrepetitive. This idea led Currie and Simpson [7] to consider sequences with a stronger property: all subsequences taken over arithmetic progressions of bounded common differences are nonrepetitive. Let $k \geq 1$ be a fixed positive integer and let $S(k)$ be the family of subsequences of S of the form $s_i s_{i+d} s_{i+2d} \dots s_{i+td}$ with $d \in \{1, 2, \dots, k\}$,

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$1 \leq i \leq k-1$, $t = \lfloor n/d \rfloor$. If every element of $S(k)$ is a nonrepetitive sequence, then S is called *nonrepetitive up to mod k* (see [7]). Let $M(k)$ denote the minimal number of symbols needed to create arbitrarily long sequences nonrepetitive up to mod k . Thue's theorem can be rephrased as $M(1) = 3$. It is easy to see that $M(k) \geq k+2$ for every $k \geq 1$, and one may suspect that equality always holds.

Conjecture 1. $M(k) = k+2$ for every $k \geq 1$.

This conjecture has been confirmed so far only for $k = 2, 3$, and 5 (it is not even known for $k = 4$) by providing Thue type constructions of the desired sequences. However, using the Lovász Local Lemma (see [2]) it was proved in [9] that $M(k) \leq e^{33}k$ for any k . In this paper we improve the last bound substantially by proving that $M(k) \leq 2k + O(\sqrt{k})$. Our method is inspired by the recent constructive proof of the Lovász Local Lemma due to Moser and Tardos [15]. Just like in the related paper [12] (and in the original nonconstructive approach), we prove the result for a more general version where symbols are chosen from prescribed lists (sets) assigned to the positions in a sequence. The same method applies in the case when K is any k -element set positive integers, and we want to construct arbitrarily long sequence with no repetitions on arithmetic subsequences with differences from K .

2. THE ALGORITHM

We present an algorithm that generates consecutive terms of a sequence S by choosing symbols at random (uniformly and independently), and every time a repetition occurs, it erases the longest repeated block and continues from the smallest unassigned position. We always erase the block that contains the last chosen element in order to ensure that after this removal the remaining sequence stays nonrepetitive. In the listing of the algorithms value 0 of some s_i means that no symbol is assigned to s_i . Initially all s_i equals 0.

We show that for any given positive integer n , and arbitrary lists of symbols L_i , each of size at least $2k + 10\sqrt{k}$, the Algorithm computes a sequence of length n which is nonrepetitive up to mod k . Random elements in line (3) of the Algorithm are chosen independently with uniform distribution. The general idea is to prove that the Algorithm cannot work forever for all possible evaluations of the random experiments. It is easy to see that the Algorithm stops only if nonrepetitive up to mod k sequence is constructed.

Theorem 1. *For every positive integer n , and for every sequence of sets L_1, \dots, L_n , each of size at least $2k + 10\sqrt{k}$, there is a sequence $S = s_1 \dots s_n$ nonrepetitive up to mod k such that $s_i \in L_i$ for every $i = 1, 2, \dots, n$.*

Proof. Let us suppose for a contradiction that such sequence does not exist. It means that the Algorithm never stops. We are going to count the possible sequences of random values used in line (3) of the algorithm in two ways.

Let r_j , $1 \leq j \leq M$, be a sequence of values chosen in the line (3) in the first M choices of some run of the Algorithm. Each r_j can take at least $10\sqrt{k}$ values. It means that there are at least $(10\sqrt{k})^{M/2}$ such sequences.

Algorithm 1 Choosing a sequence which is non-repetitive up to mod k .

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1:  $i \leftarrow 1$ 
2: while  $i \leq n$  do
3:    $s_i \leftarrow$  random element of  $L_i \setminus \{s_{i-k}, s_{i-k+1}, \dots, s_{i-1}, s_{i+1}, \dots, s_{i+k}\}$ 
4:   if  $s_1, \dots, s_n$  is non-repetitive with respect to non-zero elements then
5:      $i \leftarrow$  smallest index  $j$  for which  $s_j = 0$ 
6:   else
7:     from the set of the longest repetitions in  $S$  choose
            $s_{j-2h \cdot d+d}, \dots, s_{j-h \cdot d}, s_{j-h \cdot d+d}, \dots, s_j$ 
           with the largest index of the first element  $j - 2h \cdot d + d$ 
8:     if  $i \leq j - h \cdot d$  then
9:        $m \leftarrow j - 2h \cdot d + d$ 
10:    else
11:       $m \leftarrow j - h \cdot d + d$ 
12:    end if
13:    for  $j = 1$  to  $h$  do
14:       $s_m \leftarrow 0$ 
15:       $m \leftarrow m + d$ 
16:    end for
17:     $i \leftarrow$  smallest index  $j$  for which  $s_j = 0$ 
18:  end if
19: end while

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The second way of counting involves descriptions of the behaviour of the Algorithm. For every fixed evaluation of the first M random choices we define the following five elements:

- A route R on the upper right quadrant of a grid $\mathbb{Z} \times \mathbb{Z}$ from coordinate $(0, 0)$ to coordinate $(2M, 0)$ on $2M$ steps with possible moves $(1, 1)$ and $(1, -1)$ which never goes below the axis $y = 0$.
- A sequence D of numbers between 1 and k corresponding to the peaks on the route R , where by a *peak* we mean a move $(1, 1)$ followed immediately by a move $(1, -1)$.
- A sequence O of numbers -1 , or 1 corresponding to the peaks on route R .
- A sequence P of integers, one for every peak, whose sum is not greater than M .
- A sequence S produced by the Algorithm after M steps.

A pentad (R, D, O, P, S) will be called a *log*. We encode consecutive steps of the Algorithm into log in the following way:

Each time the algorithm executes line (3) we append a move $(1, 1)$ to the route R and for every execution of line (14) we append $(1, -1)$. Notice that in line (14) the algorithm can set zero only to s_c which are non-zero, therefore the number of down-steps on route R never excess the number of up-steps, and it never goes below axis $y = 0$. At the end of computations we add to the route R one down-step for each element of S which is non-zero. This brings us to the point $(2M, 0)$. Whenever Algorithm 1 executes line (7) we append to the sequence D a difference d of the chosen longest

repetition. Then, if (8) is true, we append 1 to the sequence O , otherwise we append -1 . For each execution of the loop (13)-(16) we append to the sequence P , value j for which m equals i in the loop. Finally, S is the sequence produced by the Algorithm after M executions of line (3).

Claim 1. *Every log corresponds to a unique sequence r_j , $1 \leq j \leq M$ of the first M values chosen in the line (3) in some execution of the Algorithm.*

Proof. For a given log (R, D, O, P, S) we are going to decode r_1, \dots, r_M . At first we use information from route R and sequences D and P to determine which s_i were non-zero at each step of the algorithm and to find coordinates of elements which were zeroed at step (14) of the Algorithm. Notice that each operation of setting a non-zero value to some s_i corresponds to the up-step $(1, 1)$ on the route R , while each zeroing of s_i corresponds to some down-step $(1, -1)$ on route R . We examine the route R from the point $(0, 0)$ to the point $(2M, 0)$. Assume that the first peak occurs after j th step. Since this is the first time we erase some elements s_i , we know that s_1, \dots, s_j are the only non-zero elements at this point. Now we use information encoded in D and P . We look at the number of consecutive down-steps on R (which in this case is equal to p_1) and remember that for this peak we zeroed $s_j, s_{j-d_1}, s_{j-2d_1}, \dots, s_{j-(p_1-1)d_1}$. Then again each up-step on R denote setting some value to the zeroed position with the smallest index i . Proceeding in that way we know exactly which position was set last, when we reach the next peak. From the number of consecutive down-steps on R we deduce the length of the zeroed repeated block. Value in the sequence D corresponding to the peak denotes the difference of the arithmetic subsequence in which the repetition occurred. Finally corresponding value from the sequence P describe the position of the symbol just set, within the erased repeated blocks. From all this information it is easy to deduce which positions was zeroed as a result of erasing the repetition. We repeat these operations until we get to the end of R .

After this preparatory step we are ready to decode r_1, \dots, r_M . We consider the sequence R in reverse order – from the last point $(2M, 0)$ to the first $(0, 0)$ modifying the final sequence S . This time we use information encoded in S and O , and the knowledge determined in a preparatory step. As we said before, each up-step $(1, 1)$ on the route R corresponds to some r_i . For every such up-step we have already determined the indices of elements r_i on S in the preparatory analysis. At the beginning, going backward on R , there is some number of down-steps corresponding to non-zero elements of S (the elements added at the end of computations). We skip them and move on. Then, each time there is an up-step on R , we assign to r_j a value from appropriate s_i (where i was determined in the preparatory step), and set s_i to 0. In fact, to determine the real outcome of random experiments (i.e. an index of the chosen element on the list of elements available at this step), we must take into account the forbidden symbols from k preceding and k following places on S . Every consecutive sequence of t down-steps on R correspond to erasure of some repeated block during the execution of the Algorithm. Then we assign to $s_i, s_{i+d_l}, \dots, s_{i+(t-1)d_l}$ corresponding values $s_{i+o_l t d_l}, s_{i+d_l+o_l t d_l}, \dots, s_{i+(t-1)d_l+o_l t d_l}$ (where s_i is the first element of the erased repeated block determined in the preparatory step). These

are exactly the values from the repetitions erased at step (17) of the Algorithm. \square

We showed that each sequence of randomly chosen values during the execution of the Algorithms corresponds to some log, and that this mapping is injective. This implies that the number of different logs is always greater than or equal to the number of feasible sequence r_1, \dots, r_M . Let L be the size of the set of all possible logs. To calculate L we have to determine the number of different structures for each element in a log. The number of all possible routes on the upper right quadrant of a grid of length $2M$ with possible moves $(1, 1)$ and $(1, -1)$ is well known to be the M th Catalan number C_M . Since in every choice in line (3) the elements occurring within the distance k are excluded, the Algorithm can not produce repeated block of length k . It means that the subsequence $(1, 1), (1, -1), (1, 1)$ can not occur in the route R . Therefore the number of peaks within R cannot exceed $M/2$. Thus there can be at most $k^{M/2}$ possible sequences D . Respectively, there are at most $2^{M/2}$ possible evaluations for sequence O .

The sequence S consists of n elements of value between 0 and $2k + 10\sqrt{k}$, which gives us $(2k + 10\sqrt{k})^n$ possible evaluations for this sequence. For every fixed route R with m peaks corresponding to the repeated block of lengths p_1, \dots, p_m we have at most $p_1 p_2 \dots p_n$ sequences which can occur as P . Therefore for the upper bound for the number sequences P we determine maximum value of the product $p_1 p_2 \dots p_n$ with $p_1 + \dots + p_n = M$. The inequality between the arithmetic and geometric means implies that the maximum is obtained when all p_i are the same. Denote their common value by x . Then we must determine $\max \left(x^{\frac{M}{x}} \right)$. Since

$$\left(x^{\frac{M}{x}} \right)' = x^{\frac{M}{x}} \left(\frac{M}{x^2} - \frac{M \log(x)}{x^2} \right),$$

we get that the maximum value is obtained with $x = e$ and equals $\approx 1.44467^M < 1.5^M$.

All these bounds brings us to the conclusion that the number of possible logs exceeds

$$\left(2k + 10\sqrt{k} \right)^n C_M k^{M/2} 2^{M/2} (1.5)^M.$$

Comparing with the number of evaluations of a sequence (r_j) we get inequality

$$(10\sqrt{k})^M \leq \left(2k + 10\sqrt{k} \right)^n C_M k^{M/2} 2^{M/2} (1.5)^M.$$

Asymptotically, Catalan numbers grow as $C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$ which implies that

$$(10\sqrt{k})^M \leq \left(2k + 10\sqrt{k} \right)^n \frac{4^M}{M\sqrt{\pi M}} k^{M/2} 2^{M/2} (1.5)^M.$$

The right hand side is $o((10\sqrt{k})^M)$ therefore for large enough M the inequality can not hold. We get a contradiction, from which we conclude that for some specific choices of r_1, r_2, \dots the algorithm stops. \square

The above proof can be applied in a more general setting.

Theorem 2. *Let K be a fixed set of k positive integers. Then for every $n \geq 1$ and for any sequence of sets L_1, \dots, L_n of size at least $2k + 10\sqrt{k}$ each, there exists a sequence $S = s_1 \dots s_n$ with $s_i \in L_i$ for all $i = 1, 2, \dots, n$ which is nonrepetitive on every arithmetic progressions whose common difference is in K .*

Note that in the proof of the Theorem 1 we focused only on the number of forbidden substructures, not their values. Given an arbitrary set of common differences we order and numerate them from 1 up to k . We can repeat the above reasoning with just one change – the sequence D consists of elements of K (but there are still k of them).

3. A RELATED GEOMETRIC PROBLEM

As stated in the introduction, the problem of finding sequences nonrepetitive up to mod k has its origin in a geometric problem of nonrepetitive coloring of points in the plane. We can apply our proof technique to a more general question in this setting. The following problem concerning nonrepetitive colorings of discrete sets of points in \mathbb{R}^n was considered in [9]. Let P be a discrete set of points and let L be a fixed set of lines in \mathbb{R}^n . A coloring of P is *nonrepetitive* (with respect to L) if each line in L is colored nonrepetitively (i.e., no sequence of consecutive points on any $l \in L$ forms a repetition). For a point $p \in P$ let $i(p)$ denote the number of lines from L incident with p and let $I = I(P, L) = \max\{i(p) : p \in P\}$ be the maximum incidence of the configuration (P, L) . Using the Lovász Local Lemma it was proved in [9] that $Ie^{(8I^2+8I-4)/(I-1)^2}$ colors are sufficient to get such a coloring. Adopting the proof of Theorem 2 we can get a better bound.

Theorem 3. *Let (P, L) be a configuration of points and lines in \mathbb{R}^n with finite maximum incidence $I > 2$. If $C \geq 2I + 10\sqrt{I}$, then there is a nonrepetitive C -coloring of P with respect to L .*

Proof. The argument is pretty much the same as in the proof of Theorem 2. We provide an algorithm for which each point is colored at random by one of $2I + 10\sqrt{I}$ colors. Fix any linear ordering of all points in P . We color them in this order using Algorithm 1, where arithmetic progressions are changed into lines in \mathbb{R}^n . Similarly, for a given point $p \in P$ and every line $l \in L$ such that $p \in l$ we forbid to use colors already assigned to I points preceding and following p on l . This gives us at most $2I$ forbidden colors for each point. So, by analogy to the previous proof, one can show that additional $10\sqrt{I}$ colors suffice to get a nonrepetitive coloring of P with respect to L . For a log (R, D, O, P, S) we take the same objects as in last case, with the exception that now D keeps the information about the line for which we get a repetition (values between 1 and I), and S is a sequence of numbers between 0 and I . Then all calculations run similarly as before. \square

4. AN OPEN PROBLEM

We would like to conclude the paper with a problem concerning infinite sets of forbidden differences. Let K be a fixed (possibly infinite) set of positive integers. A coloring of the integers is *K -nonrepetitive* if every arithmetic

progression with common difference in K forms a nonrepetitive sequence. Denote by $\pi(K)$ the minimum number of colors (possibly infinite) needed for a K -nonrepetitive coloring of \mathbb{Z} .

A natural question is for which sets K the number $\pi(K)$ is finite. An obvious necessary condition is that the related integer distance graph (i.e., a graph on the set of vertices \mathbb{Z} with two integers $a > b$ joined by an edge whenever their difference $a - b$ is in K) has finite chromatic number, denoted by $\chi(K)$. Theorem 2 shows that $\pi(K)$ is finite for finite sets K . More intriguing in this respect is the case of infinite sets K . We offer the following conjecture in the spirit of Erdős.

Conjecture 2. $\pi(K)$ is finite for every lacunary set K .

A set $K = \{k_1 < k_2 < \dots\}$ is *lacunary* if there is a real number $\delta > 0$ such that $\frac{k_{i+1}}{k_i} > 1 + \delta$ for all indices i . For instance the set of powers of 2 and the set of Fibonacci numbers are lacunary. It is known that for such sets the usual chromatic number $\chi(K)$ is finite [13], [16], [17]. However, there are non-lacunary sets with a finite chromatic number. Complete characterization of such sets is not known and, as pointed out by Ruzsa (personal communication), this problem is connected to some deep questions in additive number theory. A trivial example is the set of odd positive integers, whose chromatic number is 2. Curiously, for the nonrepetitive variant just 4 colors suffice as proved by Carpi [6], which supports even stronger supposition that perhaps $\pi(K)$ is finite if and only if $\chi(K)$ is finite.

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